

A Methodology For Formulating A Minimal Uncertainty Model For Robust Control System Design and Analysis

Christine M. Belcastro
MS 489
NASA Langley Research Center
Hampton, VA. 23665

B.-C. Chang
ME&M Dept.
Drexel University
Philadelphia, PA. 19104

Robert Fischl
ECE Dept.
Drexel University
Philadelphia, PA. 19104

Abstract

In the design and analysis of robust control systems for uncertain plants, the technique of formulating what is termed an "M- Δ model" has become widely accepted and applied in the robust control literature. The "M" represents the transfer function matrix $M(s)$ of the nominal system, and " Δ " represents an uncertainty matrix acting on $M(s)$. The uncertainty can arise from various sources, such as structured uncertainty from parameter variations or multiple unstructured uncertainties from unmodeled dynamics and other neglected phenomena. In general, Δ is a block diagonal matrix, and for real parameter variations the diagonal elements are real. As stated in the literature, this structure can always be formed for any linear interconnection of inputs, outputs, transfer functions, parameter variations, and perturbations. However, very little of the literature addresses methods for obtaining this structure, and none of this literature (to the authors' knowledge) addresses a general methodology for obtaining a minimal M- Δ model for a wide class of uncertainty. Since having a Δ matrix of minimum order would improve the efficiency of structured singular value (or multivariable stability margin) computations, a method of obtaining a minimal M- Δ model would be useful. This paper presents a generalized method of obtaining a minimal M- Δ structure for systems with real parameter variations.

1. Introduction

Robust control theory for both analysis and design has been the subject of a vast amount of research in this decade [1-35]. In particular, robust stability and performance have been emphasized in much of this work, as, for example, in the development of H^∞ control theory [10-15, 19-23]. Moreover, the development of robust control system design and analysis techniques for unstructured [1-9, 13, 19, 21] as well as structured [16-35] plant uncertainty continues to be the subject of much research - particularly the latter. Unstructured plant uncertainty arises from unmodeled dynamics and other neglected phenomena, and is complex in form. This uncertainty is called "unstructured" because it is represented as a norm-bounded perturbation with no particular assumed structure. Plant uncertainty is called "structured" when there is real parameter uncertainty in the plant model, or when there is unstructured complex uncertainty occurring in the system at multiple points simultaneously. Plant parameter uncertainty can arise from modeling errors (which usually result from assumptions and simplifications made during the modeling process and/or from the unavailability of dynamic data on which the model is based), or from parameter variations that occur during system operation.

Robust control design and analysis methods for systems with unstructured uncertainty is accomplished via singular value techniques [1-9, 21]. For systems with structured plant uncertainty, however, the structured singular value (SSV) [16-27] or multivariable stability margin (MSM) [28-33] must be used. In order to compute the SSV or MSM, the system is usually represented in terms of an M- Δ model. The "M" represents the transfer function matrix $M(s)$ of the nominal system, and " Δ " represents an uncertainty matrix acting on $M(s)$. In general, Δ is a block diagonal matrix, and for real parameter uncertainties the diagonal elements are real. As indicated in the literature [17,18,20,28], this structure can always be formed for any linear interconnection of inputs, outputs, transfer functions, parameter variations, and perturbations. However, very little of the literature discusses methods for obtaining an M- Δ model. For unstructured uncertainties, this model is very easy to obtain. However, for real parameter variations, forming an M- Δ model can be very difficult. In [29], De Gaston and Safonov present an M- Δ model for a third-order transfer function with uncertainty in the location of its two real poles and in its gain factor. Although the given M- Δ model is easily obtained for this simple example, other examples do not yield such a

straight-forward result. A general state model of $M(s)$ for additive real perturbations in the system A matrix (where A is assumed to be closed-loop) is discussed in [34]. Unfortunately, this model is not general enough for many examples, since system uncertainty is restricted to the A matrix and the uncertainty class is restricted to be linear. Morton and McAfoos [26] present a general method for obtaining an $M-\Delta$ model for linear (affine) real perturbations in the system matrices (A, B, C, D) of the open-loop plant state model. In this model, an interconnection matrix $P(s)$ is constructed first for separating the uncertainties from the nominal plant, and then $M(s)$ is formed by closing the feedback loop. The $M-\Delta$ model thus formed can be used in performing robustness analysis of a previously determined control system. If the feedback loop is not closed, μ -synthesis techniques [19-21] can be applied to the $M-\Delta$ model for robust control system design. Morton's result essentially reduces to that of [34] when the perturbations occur only in the A matrix (and the A matrix of [34] is assumed to be open-loop). An algorithm for easily computing $M(s)$ based on Morton's result is presented in [35]. Although this method of constructing an $M-\Delta$ model is general for linear uncertainties, many realistic problems require a more general class of uncertainties. Furthermore, no consideration is given to obtaining a minimal $M-\Delta$ model, where "minimal" refers to the dimension of the Δ (or M) matrix. Since the $M-\Delta$ model is a nonunique representation, it would be prudent to obtain one of minimal dimension so that the complexity of the SSV or MSM computations during robust control system design or analysis could be minimized. However, none of the literature (to the authors' knowledge) addresses the issue of minimality.

This paper presents a methodology for constructing a minimal $M-\Delta$ model for systems with real parametric multilinear uncertainties, where the term "multilinear" is defined as follows:

Definition: A function is multilinear if the functional form is linear (affine) when any variable is allowed to vary while the others remain fixed. For example, $f(a,b,c) = a + ab + bc + abc$ is a multilinear function.

Thus, the allowance of multilinear functions of the uncertain parameters provides a means of handling cross-terms in the transfer function coefficients. A procedure is proposed for obtaining this model in state-space form for uncertain single-input single-output (SISO) systems, given the system transfer function in terms of the uncertain parameters. An extension of this result to multiple-input multiple-output (MIMO) systems will be given in a subsequent publication. In this development, $M(s)$ will represent the nominal open-loop plant, so that the resulting $M-\Delta$ model may be used for robust control system analysis or design. The state-space form used in modeling $M(s)$ is an extension of Morton's result for real parametric linear (affine) uncertainties [26]. The paper is divided into the following sections. A formal statement of the problem to be solved in this paper is presented in Section 2, followed by a discussion of minimality considerations in Section 3. The approach is presented in Section 4, a proposed solution to the problem is presented in Section 5, and the proposed procedure for finding a minimal $M-\Delta$ model is summarized in Section 6. Several examples demonstrating the proposed solution are given next in Section 7, followed by some concluding remarks in Section 8.

2. Problem Statement

Given the transfer function of an uncertain system, $G(s, \delta)$, in either factored or unfactored form, as a function of the uncertain real parameters, δ , **find** a minimal $M-\Delta$ model of the form depicted below in Figure 1:

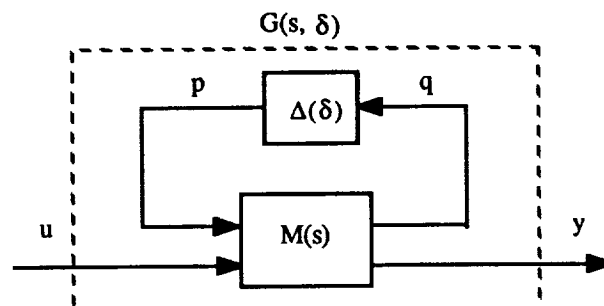


Figure 1. Block diagram of the General $M - \Delta$ Model

such that:

1. The diagonal uncertainty matrix, Δ , is of minimal dimension.
2. The model of the nominal plant, $M(s)$, is in state-space form.

The model must handle multilinear uncertainty functions in any or all of the transfer function coefficients. In order to construct a minimal M-Δ model, the dimension of the Δ matrix must be minimized. Hence, factors which have been found to affect the dimension of the M-Δ model will be discussed next, followed by the approach used in forming a solution to this problem.

3. Minimality Considerations

In constructing an M-Δ model of an uncertain system, the Δ matrix can become unnecessarily large due to repeated uncertain parameters on its main diagonal. It is therefore of interest to examine the factors which can cause this repetition, so that the number of repeated uncertain parameters can be minimized. A factor which can be shown to cause unnecessary repetition in the Δ matrix is the particular realization used in representing the system. Examples can be constructed which demonstrate this effect, and it appears that a cascade realization (and, in particular, cascaded uncertain real poles and zeros) is a desirable form for obtaining a minimal M-Δ model. Thus, a general cascade-form realization will be part of the approach taken in constructing a minimal M-Δ model. A problem arises, however, in that some transfer functions have a form which precludes cascading uncertain real poles or zeros, such as:

$$G(s, \delta) = \frac{b_1 s^2 + b_2 s + b_3}{(s + \theta_1)(s + \theta_2)}, \quad G(s, \delta) = \frac{(s + \theta_1)(s + \theta_2)}{s^2 + a_1 s + a_2}$$

where θ_1 and θ_2 are assumed here to be uncertain (and hence a function of δ). Cascading the poles and zeros for either case would result in improper transfer function blocks to be realized. For these cases, it is unavoidable for the minimal Δ matrix to have repeated uncertain parameters on the main diagonal. However, for each inseparable pole or zero pair it is only necessary to repeat one uncertain parameter. This issue will be addressed in the proposed solution, and a minimal M-Δ model for the first transfer function above will be given as an example.

Another factor which affects the dimension of the M-Δ model is the form of the coefficients in the system transfer function. If any of the coefficients is a nonlinear function of the uncertain parameters instead of a multilinear function (e.g., there are squared uncertain terms in any of the coefficients), then extra dependent uncertain parameters must be defined in order to represent these terms in a multilinear form. For example, δ_1^2 would be represented as $\delta_1 \delta_2$ where $\delta_2 = \delta_1$, and both δ_1 and δ_2 would appear in the Δ matrix. Thus, for this case, it is again necessary that the minimal Δ matrix contain repeated uncertain parameters on its main diagonal. An example illustrating this situation will be presented later.

These issues are addressed in the proposed solution for constructing a minimal M-Δ model. The approach taken in forming this solution is described in the next section.

4. Approach

Based on the problem definition and the minimality considerations outlined above, several issues will be addressed in forming a solution to the problem of constructing a minimal M-Δ model given the transfer function of an uncertain system. First, a general cascade-form realization will be found which can be used to obtain a minimal M-Δ model. Second, the minimal Δ matrix will be determined for any uncertain system such that extra dependent parameters are assigned to account for inseparable pairs of uncertain real poles or zeros as well as non-multilinear (e.g., squared) terms. Third, a method of obtaining a state-space realization of $M(s)$ for any uncertain system will be found. Therefore, the proposed approach for constructing a minimal M-Δ model is given as follows:

1. Obtain a cascade-form realization of the system so that the state-space uncertain model can be written as:

$$\begin{aligned} \dot{x} &= A x + B u \\ y &= C x + D u \end{aligned} \quad (1)$$

$$\text{where: } A = A_o + [\Delta A], \quad B = B_o + [\Delta B], \quad C = C_o + [\Delta C], \quad D = D_o + [\Delta D] \quad (2)$$

The terms with the "o" subscript (A_o, B_o, C_o, D_o) represent the nominal matrix components, and the "Δ" terms ($\Delta A, \Delta B, \Delta C, \Delta D$) represent the uncertain matrix components. To eliminate confusion of the Δ notation, the diagonal uncertainty matrix, Δ, of the M-Δ model will be represented as $[\Delta]$, and the ΔA, ΔB, ΔC, and ΔD matrices will be represented as $[\Delta A], [\Delta B], [\Delta C]$, and $[\Delta D]$ wherever clarification is required.

2. Obtain a minimal M-Δ model as described in the problem definition and pictured in Figure 1, where:

a. The minimal uncertainty matrix, Δ, is defined as:

$$\Delta = \text{diag}[\delta_1, \delta_2, \delta_3, \dots, \delta_m] = \text{diag}[\delta_I, \delta_D] = \text{diag}[\delta] \quad (3)$$

where: $\Delta \in R^{m \times m}$, $\delta_I \in R^{m_I}$, $\delta_D \in R^{m_D}$, $\delta \in R^m$

and: m = the minimal number of uncertain parameters
 m_I = the number of independent parameters given in $G(s, \delta)$
 m_D = the minimal number of dependent (or repeated) parameters

Also: $p = [\Delta] q \quad (4)$

where: p = the uncertain parameters input to $M(s)$, $p \in R^{m_p}$
 q = the uncertain variables output from $M(s)$, $q \in R^{m_q}$

Since an M-Δ model is minimal if the dimension of the Δ matrix, m , is minimal, where m depends on m_I and m_D , with m_I being given and fixed, a formal definition of a minimal M-Δ model can be stated as follows:

Definition: An M-Δ model is minimal if m_D - i.e., the number of dependent (or repeated) parameters in the Δ matrix - is minimal (or zero, if possible).

b. The state-space model of the nominal plant, $M(s)$, is an extension of Morton's result [26] and has the following form:

$$\begin{aligned} \dot{x} &= A_o x + \begin{bmatrix} B_{xp} & B_o \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix} \\ \begin{bmatrix} q \\ y \end{bmatrix} &= \begin{bmatrix} C_{qx} \\ C_o \end{bmatrix} x + \begin{bmatrix} D_{qp} & D_{qu} \\ D_{yp} & D_o \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix} \end{aligned} \quad (5)$$

where B_{xp} , C_{qx} , D_{qp} , D_{qu} , and D_{yp} are constant matrices. Thus, $M(s)$ can also be written in the equivalent shorthand notation defined as follows:

$$M(s) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} := \left[\begin{array}{c|cc} A_o & B_{xp} & B_o \\ \hline C_{qx} & D_{qp} & D_{qu} \\ C_o & D_{yp} & D_o \end{array} \right] \quad (6)$$

where: $M_{11}(s) = q(s)/p(s) = C_{qx}(sI - A_o)^{-1}B_{xp} + D_{qp}$
 $M_{12}(s) = y(s)/p(s) = C_{qx}(sI - A_o)^{-1}B_o + D_{qu}$
 $M_{21}(s) = q(s)/u(s) = C_o(sI - A_o)^{-1}B_{xp} + D_{yp}$
 $M_{22}(s) = y(s)/u(s) = C_o(sI - A_o)^{-1}B_o + D_o \quad (7)$

It should be noted that in [26] the D_{qp} matrix was required to be zero. In this paper, however, D_{qp} is allowed to be nonzero in order to model the multilinear (cross-product) uncertain terms.

The results for constructing a minimal M-Δ model via this approach are presented in the next section.

5. Proposed Solution

The proposed solution will be presented in two parts: the results for obtaining a cascade-form realization of the uncertain system will be summarized first, followed by the results for obtaining the state-space realization of a minimal M-Δ model.

5.1 Cascade-Form Realization

Given the transfer function of an uncertain system in terms of its uncertain parameters, $G(s, \delta)$, it is desired to realize the system in a cascade form of first- and second-order subsystems. Thus, if the transfer function is given in unfactored form, the numerator and denominator polynomials must be factored into first- and second-order terms. The given transfer function will then be represented as follows:

$$G(s, \delta) = K_y(\delta) G_C(s, \delta) G_R(s, \delta) K_u(\delta) \quad (8)$$

where K_u and K_y represent input and output gain terms, respectively, and G_R and G_C represent the real (first-order) and complex (second-order) transfer function components, respectively. Then :

$$G_R(s, \delta) = G_{R_k}(s, \delta) G_{R_{k-1}}(s, \delta) \dots G_{R_2}(s, \delta) G_{R_1}(s, \delta) \quad (9)$$

$$G_C(s, \delta) = G_{C_l}(s, \delta) G_{C_{l-1}}(s, \delta) \dots G_{C_2}(s, \delta) G_{C_1}(s, \delta) \quad (10)$$

$$G_{R_i}(s, \delta) = \frac{\beta_{2i-1} s + \beta_{2i}}{s + \alpha_i} \quad (11)$$

$$G_{C_i}(s, \delta) = \frac{b_{3i-2} s^2 + b_{3i-1} s + b_{3i}}{s^2 + a_{2i-1} s + a_{2i}} \quad (12)$$

and:

k = number of real (first-order) blocks

l = number of complex (second-order) blocks.

Any or all of these transfer function coefficients may be uncertain. The uncertainty may arise from either the coefficient itself being uncertain, or from the coefficient being a function of one or more uncertain variables. Therefore, for either case, any of the coefficients may be a function of δ . Furthermore, the uncertain variables may have either an additive or multiplicative form:

$$\varepsilon = \varepsilon_0 + \delta_\varepsilon \quad , \quad \varepsilon = \varepsilon_0 (1 + \delta_\varepsilon) \quad (13)$$

The following cascade-form state-space realization of this system is proposed:

$$G(s) = \left[\begin{array}{cc|c} A_R & 0 & B_R K_u \\ B_C C_R & A_C & E_C D_R K_u \\ \hline K_y D_C C_R & K_y C_C & K_y E_C D_R K_u \end{array} \right] \quad (14)$$

where:

$$A_R = \begin{bmatrix} A_{R1} & 0 & \cdots & 0 & 0 \\ B_{R2} C_{R1} & A_{R2} & \cdots & 0 & 0 \\ B_{R3} D_{R2} C_{R1} & B_{R3} C_{R2} & \cdots & 0 & 0 \\ B_{R4} D_{R3} D_{R2} C_{R1} & B_{R4} D_{R3} C_{R2} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ B_{R_{k-1}} D_{R_{k-2}} \cdots D_{R2} C_{R1} & B_{R_{k-1}} D_{R_{k-2}} \cdots D_{R3} C_{R2} \cdots & A_{R_{k-1}} & 0 \\ B_{R_k} D_{R_{k-1}} \cdots D_{R2} C_{R1} & B_{R_k} D_{R_{k-1}} \cdots D_{R3} C_{R2} \cdots & B_{R_k} C_{R_{k-1}} & A_{R_k} \end{bmatrix} \quad (15)$$

$$B_R = \begin{bmatrix} B_{R1} \\ B_{R2} D_{R1} \\ B_{R3} D_{R2} D_{R1} \\ B_{R4} D_{R3} D_{R2} D_{R1} \\ \vdots \\ B_{R_{k-1}} D_{R_{k-2}} \cdots D_{R2} D_{R1} \\ B_{R_k} D_{R_{k-1}} \cdots D_{R2} D_{R1} \end{bmatrix} \quad (16)$$

$$C_R = \begin{bmatrix} D_{R_k} D_{R_{k-1}} \cdots D_{R2} C_{R1} & D_{R_k} D_{R_{k-1}} \cdots D_{R3} C_{R2} & \cdots & D_{R_k} C_{R_{k-1}} C_{R_k} \end{bmatrix} \quad (17)$$

$$D_R = \begin{bmatrix} D_{R_k} D_{R_{k-1}} \cdots D_{R2} D_{R1} \end{bmatrix} \quad (18)$$

The A_C , B_C , C_C , and D_C matrices have the exact same form as (15) - (18), except that the subscripts "R" and "k" are replaced by "C" and "l", respectively. The submatrices are defined as follows:

$$A_{Ri} = -\alpha_i \quad B_{Ri} = 1 \quad (19)$$

$$C_{Ri} = \beta_{2i} - \alpha_i \beta_{2i-1} \quad D_{Ri} = \beta_{2i-1}$$

$$A_{Ci} = \begin{bmatrix} 0 & 1 \\ -a_{2i} & -a_{2i-1} \end{bmatrix} \quad B_{Ci} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (20)$$

$$C_{Ci} = [(b_{3i} - a_{2i} b_{3i-2}) \quad (b_{3i-1} - a_{2i-1} b_{3i-2})] \quad D_{Ci} = b_{3i-2}$$

The realizations $\{ A_{Ri}, B_{Ri}, C_{Ri}, D_{Ri} \}$ and $\{ A_{Ci}, B_{Ci}, C_{Ci}, D_{Ci} \}$ represent the i^{th} real (first-order) and complex (second-order) systems $G_{Ri}(s, \delta)$ and $G_{Ci}(s, \delta)$, respectively. Thus, for the real subsystems, $i = 1, 2, \dots, k$, and for the complex subsystems, $i = 1, 2, \dots, l$.

The resulting cascade-form realization of the uncertain system is therefore given from (14) as:

$$\begin{aligned} A &= \begin{bmatrix} A_R & 0 \\ B_C C_R & A_C \end{bmatrix} & B &= \begin{bmatrix} B_R K_u \\ B_C D_R K_u \end{bmatrix} \\ C &= \begin{bmatrix} K_y D_C C_R & K_y C_C \end{bmatrix} & D &= K_y D_C D_R K_u \end{aligned} \quad (21)$$

The above model is a general cascade-form realization for any uncertain open-loop SISO transfer function. The model does not, however, handle nonmonic denominator polynomials with uncertain leading coefficients. This would result in fractional (i.e., rational) matrix elements in the realization with uncertain parameters in the denominator of these elements. For real uncertain poles or zeros, two factors determine whether the real (first-order) or complex (second-order) block form should be used. The first is the nature of the uncertainty associated with these terms, and the second is the form of the transfer function. If the real pole or zero locations are the uncertain parameters and the transfer function form allows these poles or zeros to be separated out, then the real block form should be used. If the transfer function form does not allow this separation, then the complex block form must be used. Furthermore, if there is a pair of uncertain poles or zeros that cannot be cascaded, then the resulting minimum Δ matrix will have a repeating parameter on the main diagonal for each inseparable pole or zero pair. Alternatively, if the coefficients of the second-order polynomial associated with the real poles are the uncertain parameters, then the complex block form should be used. These cases will be illustrated in the **Examples** section of this paper. The formulation of the minimal M- Δ model will be presented next.

5.2 Minimal M- Δ Model

In formulating the minimal M- Δ model, the minimal Δ matrix must be determined first, followed by the state-space realization of M(s). Thus, the results for formulating this model will be presented in this order.

5.2.1 Minimal Δ Matrix

The minimal Δ matrix is defined as in (3) with:

$$m = m_I + m_D \quad (22)$$

where m_I is the number of independent uncertain parameters, and m_D is the number of dependent uncertain parameters that must be added. The uncertain independent parameters are those defined in $G(s, \delta)$. However, as discussed previously, the dependent uncertain parameters are those independent parameters that must be repeated due to non-multilinear terms in the transfer function coefficients and/or pairs of uncertain real poles or zeros that cannot be cascaded. Thus, for Δ to be minimal, m_D (or δ_D) should be minimized. It can be shown that if the system transfer function is formed from a given minimal M- Δ model of an uncertain system, the coefficients of the numerator and denominator polynomials will be multilinear functions of the uncertain parameters. Unfortunately, the converse is not necessarily true in general because of the dependence of the M- Δ model on the realization used for the plant. If the general cascade-form realization posed in this paper is used, however, the multilinear form of the transfer function coefficients can be used to establish that $m = m_I$ (i.e., $m_D = 0$), unless there are real uncertain pairs of poles or zeros that cannot be cascaded. Furthermore, it can be shown that if the coefficients of all the factors of the numerator and denominator polynomials are multilinear functions, then the coefficients of the expanded polynomials will also be multilinear. However, if there are non-multilinear uncertain terms in the transfer function, then dependent parameters must be defined (and added to Δ) to represent the non-multilinear term in a multilinear form. Moreover, if the non-multilinear term is of the form δ^n , then $n-1$ dependent parameters must be defined. If there are pairs of real uncertain poles or zeros that cannot be cascaded, then one additional dependent parameter must be added for each pair, and the dependent parameter can be either of the uncertain real parameters in the pair. Therefore, the number m , as determined by these rules, is the minimal dimension of the Δ matrix for the uncertainty class considered in this paper. Once this minimal dimension is determined, the Δ matrix can be defined as a diagonal matrix, as in (3), with the specified uncertain parameters on the main diagonal. Examples which illustrate these cases will be presented later in Section 6.

5.2.2 State-Space Realization of M(s)

Once the cascade-form realization has been determined, the system can be modeled as in (1) and (2), where $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$ are known functions of the uncertain parameters. Since any non-multilinear terms have

been redefined in a multilinear form when the minimal Δ matrix is determined, these matrices are multilinear functions of the parameters. In order to obtain a state-space model for $M(s)$ as defined in (5), expressions for these uncertainty matrices must be determined in terms of the matrices B_{xp} , C_{qx} , D_{qp} , D_{qu} , and D_{yp} from the model. Using (4) and (5), these expressions are determined as follows:

$$\begin{aligned} [\Delta A] &= B_{xp} [\Delta] (I - D_{qp} [\Delta])^{-1} C_{qx} = B_{xp} (I - [\Delta] D_{qp})^{-1} [\Delta] C_{qx} \\ [\Delta B] &= B_{xp} [\Delta] (I - D_{qp} [\Delta])^{-1} D_{qu} = B_{xp} (I - [\Delta] D_{qp})^{-1} [\Delta] D_{qu} \\ [\Delta C] &= D_{yp} [\Delta] (I - D_{qp} [\Delta])^{-1} C_{qx} = D_{yp} (I - [\Delta] D_{qp})^{-1} [\Delta] C_{qx} \\ [\Delta D] &= D_{yp} [\Delta] (I - D_{qp} [\Delta])^{-1} D_{qu} = D_{yp} (I - [\Delta] D_{qp})^{-1} [\Delta] D_{qu} \end{aligned} \quad (23)$$

The inverse term makes computation of D_{qp} very difficult. Furthermore, the matrix inversion can cause $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$ to have fractional (i.e., rational) elements with uncertain parameters in the denominator, which is not allowed in the uncertainty class being considered. Thus, it is desirable to represent this term in expanded form as follows:

$$(I - [\Delta] D_{qp})^{-1} = I + [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + ([\Delta] D_{qp})^3 + \dots \quad (24)$$

where the latter form in (23) has been assumed. Then the above equations can be rewritten as:

$$\begin{aligned} [\Delta A] &= B_{xp} [\Delta] C_{qx} + B_{xp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + ([\Delta] D_{qp})^3 + \dots \} [\Delta] C_{qx} \\ [\Delta B] &= B_{xp} [\Delta] D_{qu} + B_{xp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + ([\Delta] D_{qp})^3 + \dots \} [\Delta] D_{qu} \\ [\Delta C] &= D_{yp} [\Delta] C_{qx} + D_{yp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + ([\Delta] D_{qp})^3 + \dots \} [\Delta] C_{qx} \\ [\Delta D] &= D_{yp} [\Delta] D_{qu} + D_{yp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + ([\Delta] D_{qp})^3 + \dots \} [\Delta] D_{qu} \end{aligned} \quad (25)$$

The second group of terms add in the cross-terms of the multilinear uncertainty functions. Each term in the series adds a higher-order cross-product term. Since $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$ are multilinear functions with a finite number of terms, the D_{qp} matrix can be defined to have a special nilpotent structure such that:

$$D_{qp}^{r+1} = 0 \quad (26)$$

$$\text{and:} \quad (I - [\Delta] D_{qp})^{-1} = I + [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + \dots + ([\Delta] D_{qp})^r \quad (27)$$

where r is the order of the highest cross-term occurring in $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$, i.e.:

$$r = \max(O_A, O_B, O_C, O_D) \quad (28)$$

where O_A , O_B , O_C , and O_D represent the order of the highest-order cross-product term in $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$, respectively. Cross-product term order is defined as:

$$\text{order}(\delta_1 \delta_2 \delta_3 \dots \delta_i) = i - 1 \quad (29)$$

where $i = 1, 2, \dots, m$. Thus, the maximum value of r is $r_{\max} = m - 1$, where m is the dimension of the Δ matrix.

The required structure for D_{qp} to satisfy (26) and (27) is given as follows:

$$\begin{aligned} 1.) \quad d_{ii} &= 0; & i &= 1, 2, \dots, m \\ 2.) \quad \text{If } d_{ij} \neq 0, & \text{ then for } i &= 1, 2, \dots, m \text{ and } j &= 1, 2, \dots, m: \\ \quad a.) \quad d_{ji} &= 0; \\ \quad b.) \quad d_{i \oplus 1, j \oplus 1} &= 0 \text{ or } d_{i \oplus 2, j \oplus 2} = 0 \text{ or } \dots \text{ or } d_{i \oplus (m-1), j \oplus (m-1)} = 0 \end{aligned} \quad (30)$$

where the symbol " \oplus " represents "modulo m " addition. The desired equations can therefore be written as:

$$\begin{aligned}
[\Delta A] &= B_{xp} [\Delta] C_{qx} + B_{xp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + \dots + ([\Delta] D_{qp})^r \} [\Delta] C_{qx} \\
[\Delta B] &= B_{xp} [\Delta] D_{qu} + B_{xp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + \dots + ([\Delta] D_{qp})^r \} [\Delta] D_{qu} \\
[\Delta C] &= D_{yp} [\Delta] C_{qx} + D_{yp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + \dots + ([\Delta] D_{qp})^r \} [\Delta] C_{qx} \\
[\Delta D] &= D_{yp} [\Delta] D_{qu} + D_{yp} \{ [\Delta] D_{qp} + ([\Delta] D_{qp})^2 + \dots + ([\Delta] D_{qp})^r \} [\Delta] D_{qu}
\end{aligned} \quad (31)$$

where "r" is defined in (28). Since the $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$ matrices are known for the given system, the equations in (31) are used to determine B_{xp} , C_{qx} , D_{qu} , D_{yp} , and D_{qp} . Once these matrices are obtained, the state-space model of $M(s)$ is found. Hence, a minimal M - Δ model has been formed.

A procedure which summarizes the necessary steps in obtaining a minimal M - Δ model using these results is presented next.

6. Summary of Procedure

The following is a summary of the procedure implied by the above proposed approach for forming a minimal M - Δ model of a given uncertain system:

- i.) Obtain the system transfer function in factored form. The coefficients of each factor should be a multilinear function of the uncertain parameters. If necessary, define new dependent parameters to represent any non-multilinear terms in a multilinear form.
- ii.) Define the number of parameters in the Δ matrix, m , using (22). In so doing, determine if any new parameters are required to model inseparable uncertain real pole or zero pairs. If there are inseparable real pairs, either uncertain parameter in the pair may be repeated.
- iii.) Define the minimal Δ matrix as in (3), using the independent parameters defined in the given transfer function as well as those defined in steps i.) and ii.) above.
- iv.) Obtain a cascade-form realization for the system as a function of the uncertain parameters.
- v.) Express the system matrices as in (2).
- vi.) Determine the maximum order of cross-product terms, r , in $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$ as defined by (28) and (29). Then $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$ have the form represented in (31), where D_{qp} has the special (nilpotent) structure summarized by (30).
- vii.) Express $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$ as:

$$\begin{aligned}
[\Delta A] &= [\Delta A_0] + [\Delta A_1] + [\Delta A_2] + \dots + [\Delta A_r] \\
[\Delta B] &= [\Delta B_0] + [\Delta B_1] + [\Delta B_2] + \dots + [\Delta B_r] \\
[\Delta C] &= [\Delta C_0] + [\Delta C_1] + [\Delta C_2] + \dots + [\Delta C_r] \\
[\Delta D] &= [\Delta D_0] + [\Delta D_1] + [\Delta D_2] + \dots + [\Delta D_r]
\end{aligned} \quad (32)$$

where the subscript i represents the cross-terms of i^{th} order in each uncertainty matrix.

- viii.) The B_{xp} , C_{qx} , D_{yp} , and D_{qu} matrices are found using the expansion described in [26] for the uncertainty matrices having zero-order cross-product terms; i.e. define:

$$M = \begin{bmatrix} \Delta A_0 & \Delta B_0 \\ \Delta C_0 & \Delta D_0 \end{bmatrix} = M_1 \delta_1 + M_2 \delta_2 + \dots + M_m \delta_m \quad (33)$$

where the M_i matrices are appropriately partitioned. For the case of repeated parameters due to inseparable real poles or zeros, the M_i matrix associated with the repeated parameter must be nonzero. These matrices can be decomposed into the product of appropriately partitioned column and row matrices as follows:

$$M_i = \begin{bmatrix} M_{B_i} \\ M_{D_{1i}} \end{bmatrix} \begin{bmatrix} M_{C_i} & M_{D_{2i}} \end{bmatrix} \quad (34)$$

where M_{B_i} forms the i^{th} column of B_{xp} , $M_{D_{1i}}$ forms the i^{th} column of D_{yp} , M_{C_i} forms the i^{th} column of C_{qx} , and $M_{D_{2i}}$ forms the i^{th} column of D_{qu} . Thus:

$$\begin{aligned} B_{xp} &= \begin{bmatrix} M_{B_1} & M_{B_2} & \cdots & M_{B_m} \end{bmatrix} \\ D_{yp} &= \begin{bmatrix} M_{D_{11}} & M_{D_{12}} & \cdots & M_{D_{1m}} \end{bmatrix} \\ C_{qx}^T &= \begin{bmatrix} M_{C_1}^T & M_{C_2}^T & \cdots & M_{C_m}^T \end{bmatrix}^T \\ D_{qu}^T &= \begin{bmatrix} M_{D_{21}}^T & M_{D_{22}}^T & \cdots & M_{D_{2m}}^T \end{bmatrix}^T \end{aligned} \quad (35)$$

ix.) Use the higher-order cross-terms of $[\Delta A]$, $[\Delta B]$, $[\Delta C]$, and $[\Delta D]$, as in (32), to determine the elements of the D_{qp} matrix. Begin with the first-order terms and specify as many elements as possible. Continue with the second-order terms, and proceed until all elements of D_{qp} are specified. Check D_{qp} to ensure that the required special structure of (30) and, hence, (26) is satisfied.

x.) Form the minimal M- Δ model as given in (3), (5) and (6), and as pictured in Figure 1.

It should be noted that the matrices M_{B_i} , M_{C_i} , $M_{D_{1i}}$, and $M_{D_{2i}}$, obtained in decomposing the M_i matrices in (34), are not necessarily unique. A method of formalizing this decomposition for computer implementation will not be addressed in this paper. However, an algorithm is presented in [35] which accomplishes this decomposition for a more restrictive uncertainty class. Some examples will be given next to illustrate these results.

7. Examples

The following examples illustrate the proposed procedure presented above. Due to space limitations, details of each step will not be included. However, the results that are presented for each example should be fairly easily obtained.

Example 7.1

Consider the following uncertain system:

$$G(s, \delta) = \frac{(\beta_1 s + \beta_2)(\beta_3 s + \beta_4)(b_1 s^2 + b_2 s + b_3)}{(s + \alpha_1)(s + \alpha_2)(s^2 + a_1 s + a_2)}$$

where:

$$\begin{aligned} \alpha_1 &= \alpha_{1_0} + \delta\alpha_1, \quad \alpha_2 = \alpha_{2_0} + \delta\alpha_2, \quad a_1 = a_{1_0} + \delta a_1, \quad a_2 = a_{2_0} + \delta a_2 \\ \beta_1 &= \beta_{1_0} + \delta\beta_1, \quad \beta_2 = \beta_{2_0} + \delta\beta_2, \quad \beta_3 = \beta_{3_0} + \delta\beta_3, \quad \beta_4 = \beta_{4_0} + \delta\beta_4 \\ b_1 &= b_{1_0} + \delta b_1, \quad b_2 = b_{2_0} + \delta b_2, \quad b_3 = b_{3_0} + \delta b_3 \end{aligned}$$

The cascade-form realization of this example is found in a straight-forward manner to be:

$$A = \begin{bmatrix} -\alpha_1 & 0 & 0 & 0 \\ \beta_2 - \alpha_1 \beta_1 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_3(\beta_2 - \alpha_1 \beta_1) & \beta_4 - \alpha_2 \beta_3 & -a_2 & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ \beta_1 \\ 0 \\ \beta_1 \beta_3 \end{bmatrix}$$

$$C = [b_1\beta_3(\beta_2-\alpha_1\beta_1) \quad b_1(\beta_4-\alpha_2\beta_3) \quad (b_3-a_2b_1) \quad (b_2-a_1b_1)] \quad D = [\beta_1\beta_3b_1]$$

This system has eleven independent uncertain parameters. There are no non-multilinear terms in the transfer function coefficients, and there are no inseparable uncertain real pole or zero pairs. Thus, $m = 11$ and the minimal Δ matrix is defined as:

$$\Delta = \text{diag} [\delta\alpha_1, \delta\alpha_2, \delta a_1, \delta a_2, \delta\beta_1, \delta\beta_2, \delta\beta_3, \delta\beta_4, \delta b_1, \delta b_2, \delta b_3]$$

For this example, uncertainty arises in the A, B, C, and D matrices. The ΔA , ΔB , ΔC , and ΔD matrices are fairly complicated for this example, and the order of the highest cross-product term is three, i.e. $r = 3$. Following the procedure outlined in Section 6, the results are determined to be:

$$B_{xp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o}\beta_{3o} & \beta_{3o} & 1 & 1 & \beta_{3o} & \beta_{3o} & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$D_{yp} = [b_{1o}\beta_{1o}\beta_{3o} \quad b_{1o}\beta_{3o} \quad b_{1o} \quad b_{1o} \quad b_{1o}\beta_{3o} \quad b_{1o}\beta_{3o} \quad b_{1o} \quad b_{1o} \quad 1 \quad 1 \quad 1]$$

$$C_{qx}^T = \begin{bmatrix} -1 & 0 & 0 & 0 & -\alpha_{1o} & 1 & (\beta_{2o}-\alpha_{1o}\beta_{1o}) & 0 & \beta_{3o}(\beta_{2o}-\alpha_{1o}\beta_{1o}) & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -\alpha_{2o} & 1 & (\beta_{4o}-\alpha_{2o}\beta_{3o}) & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -a_{2o} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -a_{1o} & 1 & 0 \end{bmatrix}$$

$$D_{qu}^T = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ \beta_{1o} \ 0 \ \beta_{1o}\beta_{3o} \ 0 \ 0]$$

$$D_q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o} & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o}\beta_{3o} & \beta_{3o} & 1 & 1 & \beta_{3o} & \beta_{3o} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equations (5) and (6) can now be used to obtain the state-space model of $M(s)$.

Example 7.2

This example illustrates the case of an inseparable uncertain real pole pair. Consider the following system:

$$G(s, \delta) = \frac{b_1 s^2 + b_2 s + b_3}{(s + \theta_1)(s + \theta_2)}$$

where:

$$\begin{aligned} b_1 &= b_{1o} + \delta_{b_1}, \quad b_2 = b_{2o} + \delta_{b_2}, \quad b_3 = b_{3o} + \delta_{b_3} \\ \theta_1 &= \theta_{1o} + \delta_{\theta_1}, \quad \theta_2 = \theta_{2o} + \delta_{\theta_2} \end{aligned}$$

Since the numerator is second order with uncertain coefficients, the uncertain real poles in the denominator cannot be separated into the real cascade form. The denominator must therefore be expanded, and the complex (second-order) block used in the realization, which is given as follows:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\theta_1\theta_2 & -(\theta_1 + \theta_2) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= [(b_3 - \theta_1\theta_2 b_1) \quad (b_2 - (\theta_1 + \theta_2)b_1)], \quad D = [b_1] \end{aligned}$$

Since there is one inseparable uncertain pair of poles, either δ_{θ_1} or δ_{θ_2} must be repeated in the Δ matrix. (It can be shown that if this is not done, D_{qp} will not have the required structure and hence the higher-order cross-product terms will not be modeled correctly.) Since there are no non-multilinear terms in any of the transfer function coefficients, $m = 6$ (i.e., five given independent uncertain parameters plus one dependent parameter for the inseparable pole pair). The resulting Δ matrix can therefore be defined as follows:

$$\Delta = \text{diag} [\delta_{\theta_1}, \delta_{\theta_1}, \delta_{\theta_2}, \delta_{b_1}, \delta_{b_2}, \delta_{b_3}]$$

where δ_{θ_1} was arbitrarily chosen to be repeated. Using the proposed procedure, the following results are obtained:

$$B_{xp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad D_{yp} = [-b_{1o} \quad -b_{1o} \quad -b_{1o} \quad -1 \quad 1 \quad 1]$$

$$C_{qx} = \begin{bmatrix} \theta_{2o} & 0 \\ 0 & 1 \\ \theta_{1o} & 1 \\ \theta_{1o}\theta_{2o} & (\theta_{1o} + \theta_{2o}) \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D_{qu} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$D_{qp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/\theta_{2o} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 7.3

This example illustrates the case of non-multilinear terms in the transfer function. Consider the second-order uncertain system:

$$G(s, \delta) = \frac{1}{s^2 + 2\sigma s + \omega^2}, \quad \text{where:} \quad \sigma = \sigma_0 + \delta_\sigma, \quad \omega = \omega_0 + \delta_\omega$$

This is a second-order system with uncertain complex poles. The uncertainty appears in the real and imaginary components of the complex poles. The constant coefficient, ω^2 , is not a multilinear function of the uncertain parameters. Substituting for σ and ω in the above transfer function yields the following equation:

$$s^2 y = u - 2\sigma_0 s y - \omega_0^2 y - 2\delta_\sigma s y - (2\omega_0 \delta_\omega + \delta_\omega^2) y$$

In forming an M- Δ model, the problematic term is δ_ω^2 because it is not multilinear. In order to represent this equation in multilinear form, the following dependent variable is defined:

$$\delta_3 = \delta_\omega$$

so that:

$$s^2 y = u - 2\sigma_0 s y - \omega_0^2 y - 2\delta_\sigma s y - (2\omega_0 \delta_\omega + \delta_\omega \delta_3) y$$

Then the following equations can be defined:

$$\begin{aligned} q_1 &= -2 s y & p_1 &= \delta_\sigma q_1 \\ q_2 &= -y & p_2 &= \delta_\omega q_2 \\ q_3 &= -2 \omega_0 y + p_2 & p_3 &= \delta_3 q_3 \end{aligned}$$

Thus:

$$s^2 y = u - 2\sigma_0 s y - \omega_0^2 y + p_1 + p_3$$

The realization of M(s) for the resulting M- Δ model can be depicted as follows:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\sigma_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} &= \begin{bmatrix} 0 & -2 \\ -1 & 0 \\ -2\omega_0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

The Δ matrix is given by $\Delta = \text{diag} [\delta_\sigma, \delta_\omega, \delta_3]$, where $\delta_3 = \delta_\omega$.

These examples illustrate the proposed procedure for forming a minimal M- Δ model of an uncertain system. Although all the steps involved in obtaining these results have not been included, the stated results should provide a guide in performing the steps of the proposed procedure. It should be noted that, for ease of hand computation, the examples included only the simplistic (and less realistic) case in which the coefficients themselves are the uncertain parameters. However, it is emphasized that the proposed procedure does handle the more realistic case in which the uncertain transfer function coefficients are multilinear functions of the uncertain parameters.

8. Concluding Remarks

This paper has presented a proposed procedure for forming a minimal $M-\Delta$ model of an uncertain system given its transfer function in terms of the uncertain parameters. The uncertainty class considered in this paper allows the transfer function coefficients to be multilinear functions of the uncertain parameters, and the uncertainties may arise in the A, B, C, and D matrices of the system model. The proposed procedure involves realizing the system in a cascade form, determining the minimal Δ matrix of uncertain parameters, and obtaining a state-space model for the nominal system, $M(s)$. Three examples were given to illustrate the proposed procedure. The first example had eleven independent uncertain parameters, which arose in the A, B, C, and D matrices of the system realization. The second example had uncertain parameters arising in the A, C, and D matrices only. This example illustrated the formulation of a minimal $M-\Delta$ model for a system with inseparable real uncertain poles, and involved repeating an uncertain parameter in the Δ matrix. The last example had uncertainty in the A matrix only, and illustrated a method for handling non-multilinear terms.

Further work on the proposed procedure will be to include systems having a nonmonic characteristic polynomial with an uncertain leading coefficient, as well as systems having an inner feedback loop which may or may not have uncertainties. The latter case may require a modification in the formulation of the cascade realization. Although the procedure presented in this paper is for SISO systems, an extension to MIMO systems will be forthcoming, and should primarily involve modifying the cascade-form realization. Other areas of future work include development of a simple means of verifying the minimality of a given $M-\Delta$ model, and development of a method of reducing a nonminimal $M-\Delta$ model to a minimal form.

REFERENCE LIST

- [1.] J. C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis", IEEE Trans. on Aut. Contr., Vol. AC-26, pp.4-16, Feb. 1981.
- [2.] M. J. Chen and C. A. Desoer, "Necessary and Sufficient Condition for Robust Stability of Linear Distributed Feedback Systems", Int. J. Control, Vol. 35, No. 2, pp. 255-267, 1982.
- [3.] N. A. Lehtomaki, N. R. Sandell, and M. Athans, "Robustness Results in Linear-Quadratic-Gaussian Based Multivariable Control Design", IEEE Trans. on Aut. Contr., Vol. AC-26, pp. 75-92, Feb. 1981.
- [4.] N. A. Lehtomaki, D. A. Castanon, B.C. Levy, G. Stein, N. R. Sandell Jr., and M. Athans, "Robustness and Modeling Error Characterization", IEEE Trans. on Aut. Contr., Vol. AC-29, No. 3, March 1984.
- [5.] I. Postelthwaite, J. M. Edmunds, and A. G. J. MacFarlane, "Principal Gains and Principal Phases In The Analysis of Linear Multivariable Feedback Systems", IEEE Trans. on Aut. Contr., Vol. AC-26, pp. 32-46, 1981.
- [6.] M. G. Safonov, A. J. Laub, and G. L. Hartmann, "Feedback Properties of Multivariable Systems: The Role and Use of the Return Difference Matrix", IEEE Trans. on Aut. Contr., Vol. AC-26, pp. 47-65, 1981.
- [7.] J. B. Cruz, J. S. Freudenberg, and D. P. Looze, "A Relationship Between Sensitivity and Stability of Multivariable Feedback Systems", IEEE Trans. on Aut. Contr., Vol. AC-26, pp. 66-74, 1981.
- [8.] J. C. Doyle, "Robustness of Multiloop Linear Feedback Systems", 17th IEEE Conf. on Decision and Control, San Diego, USA, Jan. 1979.
- [9.] J. S. Freudenberg, D. P. Looze, and J. B. Cruz, "Robustness Analysis Using Singular Value Sensitivities", Int. J. Control, Vol. 35, pp. 95-116, 1982.
- [10.] B. A. Francis, J. W. Helton, and G. Zames, " H^∞ - Optimal Feedback Controllers for Linear Multivariable Systems", IEEE Trans. on Aut. Contr., Vol. AC-29, pp. 888-900, Oct. 1984.
- [11.] B.-C. Chang and J. B. Pearson, "Optimal Disturbance Reduction in Linear Multivariable Systems", IEEE Trans. on Aut. Contr., Vol. AC-29, pp. 880-887, Oct. 1984.
- [12.] B.-C. Chang, "A Stable State-Space Realization in the Formulation of H^∞ Norm Computation", IEEE Trans. Aut. Contr., Vol. AC-32, pp. 811-815, Sept. 1987.
- [13.] B.-C. Chang, S. S. Banda, and T. E. McQuade, "Fast Iterative Computation of Optimal Two-Block H^∞ - Norm", IEEE Trans. on Aut. Contr., Vol. AC-34, pp. 738-743, July 1989.
- [14.] B.-C. Chang, "Size Reduction in Four-Block H^∞ Formulation", Proceedings of 26th IEEE Conf. on Decision and Control, Dec. 1987.
- [15.] J. C. Doyle, K. Glover, P. Khargonekar, and B. A. Francis, "State-Space Solutions to Standard H_2 and H_∞ Control Problems", 1988 American Control Conference, Atlanta, GA, June 1988.
- [16.] M. G. Safonov and J. Doyle, "Minimizing Conservativeness of Robustness Singular Values", in S. G. Tzefestas (ed.), Multivariable Control. D. Reidel Publishing Company, 1984.

- [17.] J. Doyle, "Analysis of Feedback Systems with Structured Uncertainties", IEE Proc. Part D, Vol. 129, No. 6, pp. 242-250, 1982.
- [18.] J. C. Doyle, J. E. Wall, and G. Stein, "Performance and Robustness Analysis with Structured Uncertainty", Proceedings of the 21st IEEE Conference on Decision and Control, 1982.
- [19.] J. Doyle, "Synthesis of Robust Controllers and Filters", Proc. of 22nd IEEE Conf. on Decision and Control, Dec. 1983.
- [20.] J. Doyle, "Structured Uncertainty in Control System Design", Proc. of the 24th IEEE Conf. on Decision and Control, Dec. 1985.
- [21.] J. Doyle, Lecture Notes, ONR/Honeywell Workshop on Advances in Multivariable Control, Minneapolis, MN, Oct. 1984.
- [22.] M. G. Safonov, "Optimal H^∞ Synthesis of Robust Controllers For Systems with Structured Uncertainty", Proc. of 25th IEEE Conf. on Decision and Control, pp. 1822-1825, Dec. 1986.
- [23.] J. C. Doyle, "A Review of μ For Case Studies in Robust Control", IFAC 10th Triennial World Congress, Munich, FRG, 1987.
- [24.] M. K. H. Fan and A. L. Tits, "Characterization and Efficient Computation of the Structured Singular Value", IEEE Trans. Aut. Contr., Vol. AC-31, pp. 734-743, Aug. 1986.
- [25.] M. K. H. Fan and A. L. Tits, "m-Form Numerical Range and the Computation of the Structured Singular Value", IEEE Trans. Aut. Contr., Vol. AC-33, pp. 284-289, 1988.
- [26.] B. G. Morton and R. M. McAfoos, "A Mu-Test for Real Parameter Variations", Proc. of the 1985 American Control Conference.
- [27.] R. D. Jones, "Structured Singular Value Analysis for Real Parameter Variations", 1987 AIAA Conference on Guidance and Control.
- [28.] M. G. Safonov, "Stability Margins of Diagonally Perturbed Multivariable Feedback Systems", Proc. of the 20th IEEE Conf. on Decision and Control, Dec. 1981.
- [29.] R. E. De Gaston and M. G. Safonov, "Exact Calculation of the Multiloop Stability Margin", IEEE Trans. on Aut. Contr., Vol. AC-33, pp. 156-171, 1988.
- [30.] R. S. S. Pena and A. Sideris, "A General Program to Compute the Multivariable Stability Margin for Systems with Parametric Uncertainty", Proc. of the 1988 American Control Conference.
- [31.] A. Sideris and R. S. S. Pena, "Fast Computation of the Multivariable Stability Margin for Real Interrelated Uncertain Parameters", Proc. of the 1988 American Control Conference.
- [32.] B.-C. Chang, O. Ekdal, H. H. Yeh, and S. S. Banda, "Computation of the Real Structured Singular Value via Polytopic Polynomials", Proc. of the 1989 AIAA Guidance, Navigation and Control Conference.
- [33.] H. Bouguerra, B.-C. Chang, H. H. Yeh, and S. Banda, "A Fast Algorithm for Checking the Stability of Convex Combinations of Stable Polynomials", Technical Report, Dept. of Mechanical Engineering, Drexel University, Jan. 1989.
- [34.] G. A. Hewer, C. Kenney, and R. Klabunde, "A State Space Model of Structured Singular Values", Proc. of the 27th IEEE Conf. on Decision and Control, Dec. 1988.
- [35.] M. K. Manning and S. S. Banda, "Algorithm to Obtain M- Δ Form for Robust Control", Proc. of IEEE Conf. on Systems Engineering, Aug. 24-26, 1989, Dayton, Ohio.